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# A sharp version of Bauer–Fike's theorem

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ABSTRACT

In this paper, we present a sharp version of Bauer–Fike's theorem. We replace the matrix norm with its spectral radius or sign-complex spectral radius for diagonalizable matrices; 1-norm and $\infty$-norm for non-diagonalizable matrices. We also give the applications to the pole placement problem and the singular system.



## 1. Introduction

In this paper, we consider matrix norms $\|\cdot\|$ on the algebra $\mathbb{C}^{n\times n}$ of complex $n\times n$ matrices with the (multiplicative) unit $I$ (identity matrix), which satisfy

$$\|D\| = \max_{1\le i\le n} |d_i|$$

for all diagonal matrices $D = \mathrm{diag}(d_1, d_2, \ldots, d_n) \in \mathbb{C}^{n\times n}$. We are interested in spectral perturbation bounds for diagonalizable and non-diagonalizable elements of $\mathbb{C}^{n\times n}$. The set of all complex eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$, also known as the spectrum of $A$, is denoted by $\sigma(A)$, and we denote the spectral radius of $A$ by $\rho(A)$. For any positive integer $n$, we denote the set $\{1, 2, \ldots, n\}$ by $\langle n\rangle$.

The set of all real numbers is denoted by $\mathbb{R}$. In the linear spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, the zero vector is denoted by $\mathbf{o}$. We denote the set of all $n\times n$ matrices with real entries by $\mathbb{R}^{n\times n}$. For any $A = (a_{ij}) \in \mathbb{C}^{m\times n}$, we represent the matrix $(|a_{ij}|)$ by $|A|$. If $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{C}^{m\times n}$ and $|a_{ij}| \le |b_{ij}|$ for all $i \in \langle m\rangle$ and $j \in \langle n\rangle$, we write $|A| \le |B|$.

An interesting classical problem in perturbation theory is to investigate the relationship between the spectra of an $A \in \mathbb{C}^{n\times n}$ and a perturbation $A + E$. When $A$ is diagonalizable with $X^{-1}AX$ being diagonal, Bauer and Fike [1, Theorem III a]

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present an upper bound for the distance between a point $\mu \in \sigma(A+E)$ and $\lambda \in \sigma(A)$ is given by

$$\min_{\lambda\in\sigma(A)} |\mu-\lambda| \le \kappa(X)\|E\|, \tag{1.1}$$

where $\kappa(X) = \|X\|\,\|X^{-1}\|$ is the condition number of the matrix $X$. Other upper bounds for $|\mu-\lambda|$ is developed by Deif [2,3], Golub and Van Loan [4] (see Lemmas 2.1 and 2.2).

When $A$ is both nonsingular and diagonalizable, and $\lambda \in \sigma(A)$ is small, the estimation of the absolute error $|\mu-\lambda|$ by the upper bound in (1.1) does not provide satisfactory results, and instead, the relative error $\frac{|\mu-\lambda|}{|\lambda|}$ is being estimated. Eisenstat and Ipsen [5, Corollary 2.2] display the upper bound with $X^{-1}AX$ being diagonal,

$$\min_{\lambda\in\sigma(A)} \frac{|\mu-\lambda|}{|\lambda|} \le \kappa(X)\|A^{-1}E\| \tag{1.2}$$

for the relative error. For more related results on this topic, see [6–11].

We also consider the generalized eigenvalue problem

$$Ax = \lambda Bx. \tag{1.3}$$

If $\det(A-\lambda B) \not\equiv 0$ for $\lambda \in \mathbb{C}$ and $A, B \in \mathbb{C}^{n\times n}$, then the matrix pair $\{A, B\}$ is called regular [6,11].

Stewart [12] investigates a eigenvalue $\lambda = \alpha/\beta$ of the generalized eigenvalue problem $\beta Ax = \alpha Bx$, $(\alpha, \beta) \ne (0, 0)$ as a point in the projective complex plane $G_{12} = \{(\alpha,\beta) \ne (0,0) : \alpha, \beta \in \mathbb{C}\}$ and measures the distance between two points in the chordal metric and $\tilde{\lambda} = \tilde{\alpha}/\tilde{\beta}$, $\lambda = \alpha/\beta$,

$$\rho((\alpha,\beta),(\tilde{\alpha},\tilde{\beta})) = \frac{|\alpha\tilde{\beta} - \beta\tilde{\alpha}|}{\sqrt{|\alpha|^2+|\beta|^2}\sqrt{|\tilde{\alpha}|^2+|\tilde{\beta}|^2}} = \frac{|\lambda-\tilde{\lambda}|}{\sqrt{1+|\lambda|^2}\sqrt{1+|\tilde{\lambda}|^2}}.$$

If $\{A, B\}$, $\{C, D\}$ are regular matrix pairs, then we use it as a "distance" $d_2(Z, W)$, the "gap" between the corresponding subspaces, where $Z = (A, B)$, $W = (C, D)$. Here the "gap" is defined in the usual way as the norm of the difference of two orthogonal projectors, which are given by [11]

$$P_Z = Z^*(ZZ^*)^{-1}Z, \qquad P_W = W^*(WW^*)^{-1}W,$$

then we have the metric [11], $d_2(Z, W) = \|P_Z - P_W\|_2$.

For two regular matrix pairs $\{A, B\}$, $\{C, D\}$ with eigenvalues $(\alpha_i, \beta_i)$ and $(\gamma_i, \delta_i)$, respectively. For $Z = (A, B)$, $W = (C, D)$, we can define the generalized spectral variation of $W$ with respect to $Z$ by [11], $S_Z(W) = \max_{i\in\langle n\rangle} \min_{j\in\langle n\rangle} \rho((\alpha_i, \beta_i), (\gamma_i, \delta_i))$.

Elsner and Sun [13] extend the classical Bauer–Fike theorem to the generalized eigenvalue problem

$$\beta Ax = \alpha Bx, \qquad (A, B \in \mathbb{C}^{n\times n}, (\alpha, \beta) \ne (0, 0)).$$

**Proposition 1.1** (*[13, Theorem 2.1]*). *Let $\{A, B\}$ be a diagonalizable regular matrix pair and $Z = (A, B)$,*

$$A = P\,\mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)Q, \qquad B = P\,\mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n)Q, \ (P \text{ and } Q \text{ are invertible}).$$

*Let $\{C, D\}$ be a regular pair. Then $S_Z(W) \le \|Q^{-1}\|_2\|Q\|_2 d_2(Z, W)$, where $W = (C, D)$.*

For a generalization of the above result to regular matrix pairs by $p$-norm ($1 \le p \le \infty$) are referred to Li [10,14].

In this paper, we investigate sharp versions of Bauer–Fike's theorem. We replace the matrix norm with its spectral radius or sign-complex spectral radius for diagonalizable matrices in Section 2; 1-norm and $\infty$-norm for non-diagonalizable matrices in Section 4. We present an example for diagonalizable matrices in Section 2 and conclude with remarks in Section 6. We also improve the perturbation bound of the generalized eigenvalue problem by the sign-complex spectral radius in Section 3. We present the applications to perturbation bound of the pole placement problem and the stability for the singular system in Section 5.

## 2. Diagonalizable matrix

First we recall two important lemmas on the absolute error of eigenvalues.

**Lemma 2.1** (*Bauer–Fike [15, Theorem 6.3.2]*). *Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable with $A = X\Lambda X^{-1}$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\kappa(X) = \|X\|\,\|X^{-1}\|$. Suppose that $E \in \mathbb{C}^{n\times n}$, and let $\|\cdot\|$ be a matrix norm which satisfies $\|\mathrm{diag}(d_i)\| = \max_{i\in\langle n\rangle} |d_i|$. If $\mu \in \sigma(A+E)$, then*

$$\min_{i\in\langle n\rangle} |\mu-\lambda_i| \le \kappa(X)\|E\|.$$

A slightly different version is developed by Deif [2,3], Golub and Van Loan [4, page 328, Problem 7.2.8], respectively.

**Lemma 2.2.** *Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable with $A = X\Lambda X^{-1}$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Suppose that $E \in \mathbb{C}^{n\times n}$, and let $\|\cdot\|$ be a matrix norm. If $\mu \in \sigma(A+E)$, then*

$$\min_{i\in\langle n\rangle} |\mu - \lambda_i| \le \|\, |X^{-1}|\, |E|\, |X|\, \|.$$

There are some classical results on the spectral radius of nonnegative matrices.

**Lemma 2.3** ([15, Theorem 8.1.18]). *Let A, B and C be $n\times n$ complex matrices such that $|A| \le B$. Then*

$$\rho(AC) \le \rho(|AC|) \le \rho(|A||C|) \le \rho(B|C|).$$

Now we introduce the real spectral radius due to Rohn [16, Chapter 5] in 1989.

**Definition 2.4.** The real spectral radius of a matrix $A \in \mathbb{C}^{n\times n}$ is defined by

$$\rho_0(A) := \max\{|\lambda| : \lambda \text{ is a real eigenvalue of } A\}.$$

If $A$ has no real eigenvalues, then we set $\rho_0(A) = 0$.

A complex diagonal matrix $S = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ is called complex signature matrix, if $|s_i| = 1$ for all $i \in \langle n\rangle$. We denote by $\mathbb{CS}_n$, the set of all $n\times n$ complex signature matrices. It is obvious that $S$ is an orthogonal matrix with $|S| = I$ and $\|S\| = \max_{i\in\langle n\rangle} |s_i| = 1$ (cf. [4,17]). Especially when the entries of the diagonal matrix $S \in \mathbb{CS}_n$ are real, i.e., $s_i = \{-1, 1\}$, it is called the real signature matrix. We denote by $\mathbb{RS}_n$, the set of all $n\times n$ real signature matrices. It is clear that $\mathbb{RS}_n \subseteq \mathbb{CS}_n$.

**Definition 2.5** (*[16, Proposition 7.3], [18, Proposition 2.6]*). The sign-complex spectral radius of square matrix $A$ is defined by

$$\rho_0^{\mathbb{CS}_n}(A) = \max_{S\in\mathbb{CS}_n} \rho_0(SA).$$

**Lemma 2.6** ([19, Lemma 2.3]). *Suppose that $A \in \mathbb{C}^{n\times n}$, $\mathbf{o} \ne x \in \mathbb{C}^n$. Then $|Ax| \ge |tx| \Longrightarrow \rho_0^{\mathbb{CS}_n}(A) \ge |t|$.*

**Remark 2.7.** The real version of Lemma 2.6 can be expressed by the sign-real spectral radius (cf. [20]). If $A \in \mathbb{R}^{n\times n}$, $\mathbf{o} \ne x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$(nonnegative real numbers). Then $|Ax| \ge t|x| \Longrightarrow \rho_0^{\mathbb{RS}_n}(A) \ge t$.

Now we can develop a sharp version of Bauer–Fike's theorem with the sign-complex spectral radius.

**Theorem 2.8.** *Let $A \in \mathbb{C}^{n\times n}$ be a diagonalizable matrix with $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \mathbb{C}$ and $\mu$ is an eigenvalue of $A+E$. In addition, assume that there is an $X \in \mathbb{C}^{n\times n}$ with $X^{-1}AX = D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $X^{-1}EX \in \mathbb{C}^{n\times n}$. Then*

$$\min_{i\in\langle n\rangle} |\mu - \lambda_i| \le \rho_0^{\mathbb{CS}_n}(X^{-1}EX) \le \min\{\rho(|X^{-1}EX|), \|X^{-1}EX\|\}. \tag{2.1}$$

If, in addition, $A$ is nonsingular, then

$$\min_{i\in\langle n\rangle} \frac{|\mu - \lambda_i|}{|\lambda_i|} \le \rho_0^{\mathbb{CS}_n}(X^{-1}A^{-1}EX) \le \min\{\rho(|X^{-1}A^{-1}EX|), \|X^{-1}A^{-1}EX\|\}. \tag{2.2}$$

**Proof.** If $\mu \in \sigma(A)$, then the result follows. Assume that $\mu \notin \sigma(A)$. Since $\mu \in \sigma(A+E)$ and $X^{-1}AX = D$, we see that $\mu \in \sigma(D + X^{-1}EX)$. Then $\mu I - D - X^{-1}EX$ is singular. Thus from the invertibility of $\mu I - D$, and $X^{-1}EX \in \mathbb{C}^{n\times n}$, we see that there exists a nonzero vector $a \in \mathbb{C}^n$ such that $a = (\mu I - D)^{-1}(X^{-1}EX)a$. Thus, $|a| \le (\min_{i\in\langle n\rangle} |\mu - \lambda_i|)^{-1}|(X^{-1}EX)a|$. Then $(\min_{i\in\langle n\rangle} |\mu - \lambda_i|)|a| \le |(X^{-1}EX)a|$. It follows from Lemma 2.6 that

$$\min_{i\in\langle n\rangle} |\lambda_i - \mu| \le \rho_0^{\mathbb{CS}_n}(X^{-1}EX). \tag{2.3}$$

Let $S \in \mathbb{CS}_n$. Since $|S| = I$ and $\|S\| = 1$, we see from Lemma 2.3, Definition 2.4 and the fact that $\rho(B) \le \|B\|$ for all $B \in \mathbb{C}^{n\times n}$ that

$$\rho_0(SX^{-1}EX) \le \rho(SX^{-1}EX) \le \min\{\rho(|SX^{-1}EX|), \|SX^{-1}EX\|\} \le \min\{\rho(|X^{-1}EX|), \|X^{-1}EX\|\}.$$

Hence from the definition of $\rho_0^{\mathbb{CS}_n}(X^{-1}EX)$ and Eq. (2.3), Eq. (2.1) follows.

Now, assume, in addition, that $A$ is invertible, it follows the same approach from [5, Corollary 2.2] that we rewrite $(A+E)\hat{x} = \mu\hat{x}$ as $(\bar{A} + \bar{E})\hat{x} = \hat{x}$, where $\bar{A} = \mu A^{-1}$, $\bar{E} = -\mu A^{-1}E$.

It is obvious that 1 is an eigenvalue of $\bar{A} + \bar{E}$, the matrix $\bar{A} = \mu A^{-1}$ (which is also diagonalizable) has the same eigenvector matrix as $A$ and its eigenvalue is $\mu/\lambda_i$. Applying the result of Eq. (2.1) to $\bar{A}$ and the eigenvalue 1 of $\bar{A} + \bar{E}$, Eq. (2.2) follows. □

**Remark 2.9.** If $A$ and $E$ are real matrices, suppose $A$ is diagonalizable and $\sigma(A) \subset \mathbb{R}$, we can derive the upper bounds by the sign-real spectral radius,

$$\min_{i\in\langle n\rangle} |\mu - \lambda_i| \le \rho_0^{\mathbb{RS}_n}(X^{-1}EX) \le \min\{\rho(|X^{-1}EX|), \|X^{-1}EX\|\}. \tag{2.4}$$

Similarly, we can prove for the relative error,

$$\min_{i\in\langle n\rangle} \frac{|\mu - \lambda_i|}{|\lambda_i|} \le \rho_0^{\mathbb{RS}_n}(X^{-1}A^{-1}EX) \le \min\{\rho(|X^{-1}A^{-1}EX|), \|X^{-1}A^{-1}EX\|\}.$$

Here we will present an example to show the sharpness of our new bounds for the diagonalizable matrices.

**Example 2.10** ([2,3]). Let us consider the real matrix

$$A = \begin{pmatrix} 2 & 1\times 10^{10} & -2\times 10^{10} \\ -10^{-10} & 5 & -3 \\ 2\times 10^{-10} & -3 & 2 \end{pmatrix},$$

the eigenvalues of $A$ are $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 6$ with $X^{-1}AX = \mathrm{diag}(2, 1, 6)$ and

$$X = \begin{pmatrix} 3 & 2 & 1 \\ 2\times 10^{-10} & 2\times 10^{-10} & 2\times 10^{-10} \\ 10^{-10} & 2\times 10^{-10} & -10^{-10} \end{pmatrix}, \qquad X^{-1} = \begin{pmatrix} 0.75 & -0.5\times 10^{10} & -0.25\times 10^{10} \\ -0.5 & 0.5\times 10^{10} & 0.5\times 10^{10} \\ -0.25 & 0.5\times 10^{10} & -0.25\times 10^{10} \end{pmatrix}.$$

If we take the real perturbation $E$ such that $|E| \le 7.5\times 10^{-7} \times |A|$,

$$E = \begin{pmatrix} 1.500012500125000e-006 & -1.500005000050000e+004 & -1.499997500075000e+004 \\ 1.000005000100000e-016 & -9.999999999999997e-007 & -9.999950000999999e-007 \\ 4.999975001750000e-017 & -4.999950001500001e-007 & -4.999925000250001e-007 \end{pmatrix},$$

then we can compute $X^{-1}EX = 10^{-16} \times \begin{pmatrix} 10^5 & -1\times 10^{10} & 0 \\ 1 & 10^5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

First we compare absolute perturbation bounds of Lemmas 2.1, 2.2 and Remark 2.9, we estimate the upper bounds with respect to 2-norm, respectively,

$$\begin{aligned} \kappa(X)\|E\|_2 = 7.4246\times 10^{14} &\gg \|\,|X^{-1}|\,|E|\,|X|\,\|_2 = 2.3569\times 10^{-5} \\ &> \|X^{-1}EX\|_2 = \|\,|X^{-1}EX|\,\|_2 = 1.0000\times 10^{-6} \\ &\gg \rho(|X^{-1}EX|) = 2\times 10^{-11} \\ &> \rho_0^{\mathbb{RS}_3}(X^{-1}EX) = \sqrt{2}\times 10^{-11}. \end{aligned}$$

Second we study relative perturbation bounds of Eq. (1.2), Remark 2.9, and the upper bound with 2-norm holds,

$$\begin{aligned} \kappa(X)\|A^{-1}E\|_2 = 3.7123\times 10^{14} &\gg \|X^{-1}A^{-1}EX\|_2 = 5.0000\times 10^{-7} \\ &\gg \rho(|X^{-1}A^{-1}EX|) = 1.5000\times 10^{-11} \\ &> \rho_0^{\mathbb{RS}_3}(X^{-1}A^{-1}EX) = 1.2808\times 10^{-11}. \end{aligned}$$

The computed eigenvalues of $A + E$ are

$$\hat{\lambda}_1 = 2 + 1.0000\times 10^{-11}, \qquad \hat{\lambda}_2 = 1 + 1.0000\times 10^{-11}, \qquad \hat{\lambda}_3 = 6 + 10^{-16}.$$

We can obtain

$$|\lambda_1 - \hat{\lambda}_1| = 1.0000\times 10^{-11}, \qquad |\lambda_2 - \hat{\lambda}_2| = 1.0000\times 10^{-11}, \qquad |\lambda_3 - \hat{\lambda}_3| = 1\times 10^{-16};$$

and

$$\frac{|\lambda_1 - \hat{\lambda}_1|}{|\lambda_1|} = 5.0000\times 10^{-12}, \qquad \frac{|\lambda_2 - \hat{\lambda}_2|}{|\lambda_2|} = 1.0000\times 10^{-11}, \qquad \frac{|\lambda_3 - \hat{\lambda}_3|}{|\lambda_3|} = \frac{1}{6}\times 10^{-17}.$$

Our perturbation bounds are sharper than the known results.

## 3. Perturbation bound for the generalized eigenvalue problem

In this section, we present a sharp version of Bauer–Fike's theorem to the generalized eigenvalue problem.

### 3.1. Either the matrix A or B is invertible

Assume that we have a diagonal matrix $\tilde{D} \in \mathbb{C}^{n\times n}$ and a matrix $\tilde{X} \in \mathbb{C}^{n\times n}$ such that $A\tilde{X} \approx B\tilde{X}\tilde{D}$, which means $A\tilde{x}_i \approx \tilde{\lambda}_i B\tilde{x}_i$ for all $i \in \langle n\rangle$, where $\tilde{\lambda}_i$ and $\tilde{x}_i$ denote the $(i, i)$-element of $\tilde{D}$ and the $i$-th column of $\tilde{X}$, respectively.

**Theorem 3.1.** *Let $Y$ be an arbitrary $n \times n$ complex matrix and $R_1, R_2$ be defined as follows,*

$$R_1 := Y(A\tilde{X} - B\tilde{X}\tilde{D}), \qquad R_2 := YB\tilde{X} - I.$$

*If $\rho(R_2) < 1$, then $B, \tilde{X}$ and $Y$ are nonsingular, and*

$$\min_{i\in\langle n\rangle} |\lambda - \tilde{\lambda}_i| \le \rho_0^{\mathbb{CS}_n}((I + R_2)^{-1}R_1) \le \min\{\rho(|(I + R_2)^{-1}R_1|), \|(I + R_2)^{-1}R_1\|\} \le \frac{\|R_1\|}{1 - \|R_2\|}. \tag{3.1}$$

**Proof.** From $\rho(R_2) < 1$ and $YB\tilde{X} = I + R_2$, it is obvious that $B, \tilde{X}$ and $Y$ are nonsingular. Since $B$ is invertible, Eq. (1.3) is equivalent to the standard eigenvalue problems $B^{-1}Ax = \lambda x$, which implies that $B^{-1}A - \lambda I$ is singular.

If $\tilde{\lambda}_i \in \sigma(\tilde{D})$, then the result follows. Assume that $\tilde{\lambda}_i \notin \sigma(\tilde{D})$, in this case $\tilde{D} - \lambda I$ is nonsingular. Then it holds that

$$\begin{aligned}(\tilde{D} - \lambda I)^{-1}\tilde{X}^{-1}(B^{-1}A - \lambda I)\tilde{X} &= [(\tilde{D} - \lambda I)^{-1}\tilde{X}^{-1}][\tilde{X}(\tilde{D} - \lambda I) - \tilde{X}(\tilde{D} - \lambda I) + (B^{-1}A - \lambda I)\tilde{X}] \\ &= I - (\tilde{D} - \lambda I)^{-1}\tilde{X}^{-1}B^{-1}(B\tilde{X}\tilde{D} - A\tilde{X}).\end{aligned} \tag{3.2}$$

From the singularity of Eq. (3.2) and the invertibility of $\tilde{D} - \lambda I$, we see that there exists a nonzero vector $b \in \mathbb{C}$, such that $b = (\tilde{D} - \lambda I)^{-1}\tilde{X}^{-1}B^{-1}(B\tilde{X}\tilde{D} - A\tilde{X})b$. Thus,

$$|b| \le \left(\min_{i\in\langle n\rangle} |\tilde{\lambda}_i - \lambda|\right)^{-1} |\tilde{X}^{-1}B^{-1}(B\tilde{X}\tilde{D} - A\tilde{X})b|.$$

Then $(\min_{i\in\langle n\rangle} |\tilde{\lambda}_i - \lambda|)|b| \le |\tilde{X}^{-1}B^{-1}(B\tilde{X}\tilde{D} - A\tilde{X})b|$. It follows from Lemma 2.6 that

$$\min_{i\in\langle n\rangle} |\lambda_i - \mu| \le \rho_0^{\mathbb{CS}_n}(\tilde{X}^{-1}B^{-1}(B\tilde{X}\tilde{D} - A\tilde{X})). \tag{3.3}$$

It is easy to show that

$$\tilde{X}^{-1}B^{-1}(B\tilde{X}\tilde{D} - A\tilde{X}) = -[I + (YB\tilde{X} - I)]^{-1}Y(A\tilde{X} - B\tilde{X}\tilde{D}) = -(I + R_2)^{-1}R_1. \tag{3.4}$$

Since $\|(I + R_2)^{-1}R_1\| \le \|(I + R_2)^{-1}\| \, \|R_1\| \le \frac{\|R_1\|}{1-\|R_2\|}$, and from Eqs. (3.3) and (3.4), Eq. (3.1) follows. □

**Remark 3.2.** On comparing Theorem 3.1 with [21, Theorem 1], it is obvious that the condition of Theorem 3.1 is weaker than that of [21, Theorem 1], but our result is stronger.

### 3.2. Diagonalizable regular matrix pair

In this subsection, we derive a sharp version of Bauer–Fike's theorem of the diagonalizable regular matrix pair.

**Definition 3.3** (*[11]*)**.** A regular matrix pair $\{A, B\}$ is called diagonalizable, if there exist nonsingular matrices $P$ and $Q$ such that for any $\mu \in \mathbb{C}$, $\mu B - A = P(\mu I - D)Q$, where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\lambda_i$ is eigenvalue of a regular matrix pair $\{A, B\}$.

Now we develop a sharp version of Bauer–Fike's theorem for the diagonalizable regular matrix pair.

**Theorem 3.4.** *Let $\{A, B\}$ be a diagonalizable regular matrix pair with $\sigma(\{A, B\}) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \mathbb{C}$ and $\mu$ is eigenvalue of $(A + \delta A)y = \mu(B + \delta B)y$. In addition, assume that there are nonsingular matrices $P$ and $Q$ with $\mu B - A = P(\mu I - D)Q$. Then*

$$\min_{i\in\langle n\rangle} |\lambda_i - \mu| \le \rho_0^{\mathbb{CS}_n}(P^{-1}EQ^{-1}),$$

*where $E = \mu\delta B - \delta A$.*

**Proof.** If $\mu \in \sigma(\{A, B\})$, then the result follows. Assume that $\mu \notin \sigma(\{A, B\})$. Since $\mu \in \sigma(\{A + \delta A, B + \delta B\})$ and $P^{-1}(\mu B - A)Q^{-1} = \mu I - D$, we see that $\mu \in \sigma(D + P^{-1}EQ^{-1})$. Then $\mu I - D - P^{-1}EQ^{-1}$ is singular. From the invertibility of $\mu I - D$ and $P^{-1}EQ^{-1} \in \mathbb{C}^{n\times n}$, there exists a nonzero vector $z \in \mathbb{C}^n$ such that $z = (\mu I - D)^{-1}(P^{-1}EQ^{-1})z$. Thus $|z| \le (\min_{i\in\langle n\rangle} |\mu - \lambda_i|)^{-1}|(P^{-1}EQ^{-1})z|$. Then $(\min_{i\in\langle n\rangle} |\mu - \lambda_i|)|z| \le |(P^{-1}EQ^{-1})z|$. It follows from Lemma 2.6 that

$$\min_{i\in\langle n\rangle} |\lambda_i - \mu| \le \rho_0^{\mathbb{CS}_n}(P^{-1}EQ^{-1}). \quad \square$$

## 4. Non-diagonalizable case

It is a position to develop another version of Bauer–Fike's theorem of the non-diagonalizable case [22,23].

Suppose $\mathscr{C}_n$ is the set of $n \times n$ column stochastic matrices, $\mathscr{R}_n$ is the set of $n \times n$ row stochastic matrices, and $\mathscr{E}_n$ is the set of $n \times n$ nonnegative matrices such that each column consists of exactly one entry with value one and all other entries zero, $\mathscr{F}_n$ is the set of $n \times n$ nonnegative matrices such that each row consists of exactly one entry with value one and all other entries zero. We recall a useful lemma on nonnegative matrices [24].

**Lemma 4.1** (*[24, Corollary 3.2]*). *Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. Then there exists a matrix $E$ in $\mathscr{E}_n$ such that $\rho(EA) = \max_{C\in\mathscr{C}_n} \rho(CA)$.*

**Corollary 4.2.** *Let $B \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. Then there exists a matrix $F$ in $\mathscr{F}_n$ such that $\rho(BF) = \max_{R\in\mathscr{R}_n} \rho(BR)$.*

**Proof.** Let $B = A^T$, there exists a matrix $F^T \in \mathscr{E}_n$, we have

$$\rho(BF) = \rho(F^TA) = \max_{C\in\mathscr{C}_n} \rho(CA) = \max_{C^T\in\mathscr{R}_n} \rho(BC^T) = \max_{R\in\mathscr{R}_n} \rho(BR). \quad \square$$

Now we present a sharp version of Bauer–Fike's theorem for the non-diagonalizable matrices with 1-norm. Assume the perturbation of the generalized eigenvalue problem of $Ax = \lambda Bx$ is

$$(A + \delta A)y = \mu(B + \delta B)y.$$

For the regular matrix pair $\{A, B\}$, we consider the Kronecker canonical form [25, page 264, Chapter 8.7.2] of $\mu B - A$

$$P^{-1}(\mu B - A)Q^{-1} \equiv J = \mathrm{diag}(M_1, M_2, \ldots, M_b), \tag{4.1}$$

where $P$ and $Q$ are nonsingular matrices and $M_i = A_i - \mu B_i$ must be either $J(\lambda_i)$ $(i = 1, 2, \ldots, s)$ or $N_i$ $(i = s+1, s+2, \ldots, b)$.

$$J(\lambda_i) = \begin{pmatrix} \lambda_i - \mu & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i - \mu \end{pmatrix} \quad \text{and} \quad N_i = \begin{pmatrix} 1 & \mu & & \\ & \ddots & \ddots & \\ & & \ddots & \mu \\ & & & 1 \end{pmatrix}. \tag{4.2}$$

$J(\lambda_i)$ is a block corresponding to finite eigenvalue $\lambda_i$ and $N_i$ is a block corresponding to the infinite eigenvalue.

Next we present another version of the non-diagonalizable Bauer–Fike theorem for the $n$-by-$n$ generalized eigenvalue problem $Ax = \lambda Bx$.

**Theorem 4.3.** *Let the regular matrix pair $\{A, B\}$ with $\mu B - A = P\,\mathrm{diag}(J(\lambda_1), J(\lambda_2), \ldots, J(\lambda_s), N_{s+1}, N_{s+2}, \ldots, N_b)Q = \mathrm{diag}(M_1, M_2, \ldots, M_b)$. The size of block $M_i$ is $m_i \times m_i$ $(i = 1, 2, \ldots, b)$ and $m_1 + m_2 + \cdots + m_b = n$. Suppose that $\delta A, \delta B \in \mathbb{C}^{n\times n}$ and $\mu \in \sigma(\{A + \delta A, B + \delta B\})$. Then*

$$\min_{i\in\langle b\rangle} \sigma_{i,m_i}^{-1} \le \rho(|P^{-1}EQ^{-1}|T) \le \|P^{-1}EQ^{-1}\|_1, \tag{4.3}$$

*where $T \in \mathscr{E}_m$, $E = \mu\delta B - \delta A$ and* $\sigma_{ij} = \begin{cases} \sum_{k=1}^{j} |\lambda_i - \mu|^{-k}, & \text{if } i = 1, 2, \ldots, s \\ \sum_{k=0}^{j-1} |\mu|^k, & \text{if } i = s+1, s+2, \ldots, b. \end{cases}$

**Proof.** If $\mu \in \sigma(\{A, B\})$, then the result follows. Assume that $\mu \notin \sigma(\{A, B\})$. Since $\mu \in \sigma(\{A + \delta A, B + \delta B\})$ and $P^{-1}(\mu B - A)Q^{-1} \equiv J$, we see that $\mu \in \sigma(J + P^{-1}EQ^{-1})$. Then $\mu I - J - P^{-1}EQ^{-1}$ is singular. From the invertibility of $\mu I - J$, we deduce that $I - (\mu I - J)^{-1}(P^{-1}EQ^{-1})$ is singular. Thus

$$\rho((\mu I - J)^{-1}(P^{-1}EQ^{-1})) \ge 1. \tag{4.4}$$

Next, let us denote that $|(\mu I - J)^{-1}| \equiv \mathrm{diag}(J_1, J_2, \ldots, J_b)$. Then we have

$$J_i = J(\lambda_i) = \left| \begin{pmatrix} \lambda_i - \mu & -1 & & \\ & \lambda_i - \mu & \ddots & \\ & & \ddots & -1 \\ & & & \lambda_i - \mu \end{pmatrix}^{-1} \right| = \begin{pmatrix} |\lambda_i - \mu|^{-1} & |\lambda_i - \mu|^{-2} & \cdots & |\lambda_i - \mu|^{-m_i} \\ & |\lambda_i - \mu|^{-1} & \ddots & \vdots \\ & & \ddots & |\lambda_i - \mu|^{-2} \\ & & & |\lambda_i - \mu|^{-1} \end{pmatrix}$$

$(i = 1, 2, \ldots, s)$,

and

$$J_i = N_i = \left| \begin{pmatrix} 1 & \mu & & \\ & 1 & \ddots & \\ & & \ddots & \mu \\ & & & 1 \end{pmatrix}^{-1} \right| = \begin{pmatrix} 1 & |\mu| & \cdots & |\mu|^{m_i-1} \\ & 1 & \ddots & \vdots \\ & & \ddots & |\mu| \\ & & & 1 \end{pmatrix} \quad (i = s+1, s+2, \ldots, b).$$

Suppose $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2, \ldots, \Sigma_b)$, where $\Sigma_i = \operatorname{diag}(\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{i,m_i}) \in \mathbb{R}^{m_i \times m_i}$. We can rewrite

$$|(\mu I - J)^{-1}| = \operatorname{diag}(S_1, S_2, \ldots, S_b)\Sigma,$$

where $S_i$ is an $m_i \times m_i$ column stochastic matrix, i.e.,

$$S_i = \begin{pmatrix} 1 & \frac{|\lambda_i-\mu|^{-2}}{\sigma_{i2}} & \frac{|\lambda_i-\mu|^{-3}}{\sigma_{i3}} & \cdots & \frac{|\lambda_i-\mu|^{-m_i}}{\sigma_{i,m_i}} \\ & \frac{|\lambda_i-\mu|^{-1}}{\sigma_{i2}} & \frac{|\lambda_i-\mu|^{-2}}{\sigma_{i3}} & \cdots & \frac{|\lambda_i-\mu|^{-m_i+1}}{\sigma_{i,m_i}} \\ & & \frac{|\lambda_i-\mu|^{-1}}{\sigma_{i3}} & \cdots & \frac{|\lambda_i-\mu|^{-m_i+2}}{\sigma_{i,m_i}} \\ & & & \ddots & \vdots \\ & & & & \frac{|\lambda_i-\mu|^{-1}}{\sigma_{i,m_i}} \end{pmatrix} \in \mathscr{C}_m.$$

Now we apply Lemma 2.3 to Eq. (4.4), (there exists a matrix $T \in \mathscr{E}_m$)

$$\begin{aligned} 1 &\le \rho[(\mu I - J)^{-1}(P^{-1}EQ^{-1})] \le \rho[|(\mu I - J)^{-1}|\,|P^{-1}EQ^{-1}|] \\ &= \rho\,(\operatorname{diag}(S_1, S_2, \ldots, S_b)\Sigma|P^{-1}EQ^{-1}|) \le \rho(T\Sigma|P^{-1}EQ^{-1}|) = \rho(\Sigma|P^{-1}EQ^{-1}|T) \\ &\le \max_{i\in\langle b\rangle}\left(\max_{j\in\langle m_i\rangle}\sigma_{ij}\right)\rho(|P^{-1}EQ^{-1}|T) = \max_{i\in\langle b\rangle}\sigma_{i,m_i}\rho(|P^{-1}EQ^{-1}|T). \end{aligned}$$

Thus $\min_{i\in\langle b\rangle}\sigma_{i,m_i}^{-1} \le \rho(|P^{-1}EQ^{-1}|T) \le \|P^{-1}EQ^{-1}\|_1$. $\square$

Next we try to make the left side of Eq. (4.3) more clearly. Since $\mu \notin \sigma(\{A, B\})$, and we suppose the matrix pair $\{A+\delta A, B+\delta B\}$ are also regular, then $\mu$ is finite. In this case, when $\lambda_i$ is infinite, we cannot estimate the bound of $\lambda_i - \mu$. We only consider the case when $\lambda_i$ is finite.

We assume that the perturbation matrices $\delta A$ and $\delta B$ is not large, then $\min_{i\in\langle n\rangle}|\lambda_i - \mu|$ is small enough such that $\min_{i\in\langle n\rangle}|\lambda_i - \mu|^{-1} \gg |\mu|$, then $(\sum_{k=1}^{j}|\lambda_i-\mu|^{-k}) \gg (\sum_{k=0}^{j-1}|\mu|^k)$. Suppose the largest size of the block $J_i$ is $m \times m$, we can get a more simple result from Eq. (4.3) that

$$\begin{aligned} \|P^{-1}EQ^{-1}\|_1 &\ge \rho(|P^{-1}EQ^{-1}|T) \ge \min_{i\in\langle b\rangle}\sigma_{i,m_i}^{-1} \ge \min_{i\in\langle n\rangle}\left(\sum_{i=1}^{m_i}|\lambda_i-\mu|^{-i}\right)^{-1} \ge \min_{i\in\langle n\rangle}\left(\sum_{i=1}^{m}|\lambda_i-\mu|^{-i}\right)^{-1} \\ &= \min_{i\in\langle n\rangle}\frac{|\lambda_i-\mu|^m}{1+|\lambda_i-\mu|+|\lambda_i-\mu|^2+\cdots+|\lambda_i-\mu|^{m-1}}. \end{aligned}$$

In a conclusion, we can obtain a useful corollary.

**Corollary 4.4.** *Suppose $\lambda_i$ is an eigenvalue of regular matrix pair $\{A, B\}$ and $\mu$ is an eigenvalue of regular matrix pair $\{A+\delta A, B+\delta B\}$. There exist nonsingular matrices P and Q such that*

$$\mu B - A = P\operatorname{diag}(J(\lambda_1), J(\lambda_2), \ldots, J(\lambda_s), N_{s+1}, N_{s+2}, \ldots, N_b)Q.$$

*Let the largest size of $J_i$ be $m \times m$, then there exists a matrix $T \in \mathscr{E}_m$, such that*

$$\min_{i\in\langle n\rangle}\frac{|\lambda_i-\mu|^m}{1+|\lambda_i-\mu|+|\lambda_i-\mu|^2+\cdots+|\lambda_i-\mu|^{m-1}} \le \rho(|P^{-1}EQ^{-1}|T) \le \|P^{-1}EQ^{-1}\|_1, \tag{4.5}$$

*where $E = \mu\delta B - \delta A$.*

Similarly, we can obtain the upper bound with $\infty$-norm.

**Corollary 4.5.** *Suppose $\lambda_i$ is an eigenvalue of regular matrix pair $\{A, B\}$ and $\mu$ is an eigenvalue of regular matrix pair $\{A+\delta A, B+\delta B\}$. There exist nonsingular matrix P and Q such that*

$$\mu B - A = P\,\mathrm{diag}(J(\lambda_1), J(\lambda_2), \ldots, J(\lambda_s), N_{s+1}, N_{s+2}, \ldots, N_b)Q.$$

*Let the largest size of $J_i$ be $m \times m$, then there exists a matrix $W \in \mathscr{F}_m$, such that*

$$\min_{i\in\langle n\rangle} \frac{|\lambda_i - \mu|^m}{1 + |\lambda_i - \mu| + |\lambda_i - \mu|^2 + \cdots + |\lambda_i - \mu|^{m-1}} \le \rho(W|P^{-1}EQ^{-1}|) \le \|P^{-1}EQ^{-1}\|_\infty, \quad (E = \mu\delta B - \delta A). \tag{4.6}$$

It follows from [14, Lemma 3.1] that we can deduce the corollary.

**Corollary 4.6.** *If $\|P^{-1}EQ^{-1}\|_p \le \frac{1}{m}$, $(p = 1, \infty)$ then*

$$s \le \begin{cases} \|P^{-1}EQ^{-1}\|_p^{\frac{1}{n}} + \|P^{-1}EQ^{-1}\|_p^{\frac{2}{n}}, & \text{if } \|P^{-1}EQ^{-1}\|_p \le 1, \\ \|P^{-1}EQ^{-1}\|_p^{\frac{1}{n}} + \|P^{-1}EQ^{-1}\|_p, & \text{if } \|P^{-1}EQ^{-1}\|_p > 1, \end{cases}$$

*where $s = \max_{j\in\langle m\rangle}\{\min_{i\in\langle m\rangle} |\lambda_i - \mu_j|\}$, $\lambda_i \in \sigma(\{A, B\})$ and $\mu_j \in \sigma(\{A+\delta A, B+\delta B\})$.*

**Corollary 4.7.** *If $\mu$ is any eigenvalue of $B = A + E$, and $A = XJX^{-1}$ is Jordan canonical form of A, m is the largest size of Jordan block, then*

$$\min_{i\in\langle n\rangle} \frac{|\lambda_i - \mu|^m}{1 + |\lambda_i - \mu| + |\lambda_i - \mu|^2 + \cdots + |\lambda_i - \mu|^{m-1}} \le \|X^{-1}EX\|_p, \quad (p = 1, \infty).$$

**Proof.** It is the special case of Eqs. (4.5) and (4.6) when $B = I$ and $\delta B = 0$. □

**Remark 4.8** (*[26]*)**.** If $\mu$ is any eigenvalue of $B = A + E \in \mathbb{R}^{n\times n}$, and $A = XJX^{-1}$ is the Jordan canonical form of $A$, then there exists an eigenvalue $\lambda \in \sigma(A)$ such that

$$\frac{|\mu - \lambda|^m}{(1 + |\mu - \lambda|)^{m-1}} \le \|X^{-1}EX\|_2,$$

where $m$ is the largest Jordan block with respect to $\lambda$. If $\|X^{-1}EX\|_2 \le \frac{1}{2^{m-1}}$, then $\min_{i\in\langle n\rangle} |\lambda_i - \mu| \le 2^{1-\frac{1}{m}}\|X^{-1}EX\|_2^{\frac{1}{m}}$.

Stewart and Sun [11, page 174, Theorem 1.12] present a sharp bound with respect to 2-norm,

$$\frac{|\lambda - \mu|^m}{1 + |\lambda - \mu| + |\lambda - \mu|^2 + \cdots + |\lambda - \mu|^{m-1}} \le \|X^{-1}EX\|_2.$$

## 5. Applications to the pole placement problem and the singular system

In this section, we give the application of sharp versions of Bauer–Fike's theorem to the pole placement problem [27,28] and the singular system.

**Proposition 5.1** (*[27]*)**.** ***Single-input pole placement (SIPP)*** *Given a set of n numbers $\mathscr{P} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, find a vector $f$, such that the spectrum of $A - bf^T$ is equal to $\mathscr{P}$.*

It has been proved that [27]: the vector $f$ exists for all set $\mathscr{P}$ if and only if $(A, b)$ is controllable, i.e.,

$$\mathrm{rank}[b, A - \lambda I] = n, \quad \forall\lambda.$$

It is easy to know from the fact that if $A - bf^T$ is non-diagonalizable, it has two different eigenvalues corresponding to the same eigenvalue $\lambda$, then $\mathrm{rank}[A - bf^T - \lambda I] \le n - 2$, and thus $\mathrm{rank}[b, A - bf^T - \lambda I] \le n - 1$ which contradicts the controllability of $(A, b)$. So we could denote $A - bf^T = GDG^{-1}$ and $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a diagonal matrix.

**Lemma 5.2** (*[27, Theorem 3.3]*)**.** *Consider the **SIPP** problem with data A, b, $\mathscr{P} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and consider a perturbed problem with data $\hat{A} := A + \delta A$, $\hat{b} := b + \delta b$, $\hat{\mathscr{P}} := \{\lambda_1 + \delta\lambda_1, \lambda_2 + \delta\lambda_2, \ldots, \lambda_n + \delta\lambda_n\}$. Assume that the desired poles $\lambda_j$, and the perturbed poles $\lambda_j + \delta\lambda_j$, $(j = 1, 2, \ldots, n)$ are each pairwise different.*

*Suppose further that $\|\delta A\|, \|\delta b\|, \|\delta\lambda\| \le \epsilon$ for sufficiently small $\epsilon$. Let $f, \hat{f}$ be the feedback gains of the unperturbed and the perturbed system, respectively, and let $\hat{\kappa}$ be the spectral condition number of the perturbed closed loop system $\hat{A} - \hat{b}\hat{f}^T$ and its spectral set is $\{\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_n\}$. Then for each of the eigenvalues $\mu$ of $A - b\hat{f}^T$, there is a pole $\lambda_i$ of the unperturbed closed loop system $A - bf^T$ such that*

$$|\lambda_i - \mu| \le \epsilon[1 + (1 + \|\hat{f}\|\hat{\kappa})]. \tag{5.1}$$

We can improve the above result as follows.

**Theorem 5.3.** *With the same condition of Lemma 5.2, suppose* $\hat{A} - \hat{b}\hat{f}^T = \hat{G}\hat{D}\hat{G}^{-1}$, $\hat{D} = \mathrm{diag}(\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_n)$ *and* $\hat{\kappa} = \|\hat{G}\|_2\|\hat{G}^{-1}\|_2$, *we can get*

$$|\lambda_i - \mu| \le \rho_0^{\mathbb{CS}_n}[\hat{G}(\delta A - \delta b\hat{f})\hat{G}^{-1}] + \epsilon \le \epsilon[1 + (1 + \|\hat{f}\|\hat{\kappa})]. \tag{5.2}$$

**Proof.** Since $A - b\hat{f}^T = \hat{A} - \hat{b}\hat{f}^T - \delta A + \delta b\hat{f}^T$, we can obtain

$$\begin{aligned} |\mu - \lambda_i| &\le |\mu - \hat{\lambda}_i| + |\delta\lambda_i| \le \rho_0^{\mathbb{CS}_n}[\hat{G}(\delta A - \delta b\hat{f})\hat{G}^{-1}] + \epsilon \\ &\le \|\hat{G}(\delta A - \delta b\hat{f})\hat{G}^{-1}\| + \epsilon \le \hat{\kappa}\|\delta A - \delta b\hat{f}\| + \epsilon \\ &\le \epsilon[1 + (1 + \|\hat{f}\|\hat{\kappa})]. \quad \square \end{aligned}$$

Similarly, we can present a sharp bound of multi-input pole placement (MIPP) problem of [29].

$$|\lambda_i - \mu| \le \rho_0^{\mathbb{CS}_n}[\hat{G}(\delta A - \delta B\hat{F})\hat{G}^{-1}] + \epsilon \le \epsilon[1 + (1 + \|\hat{F}\|\hat{\kappa})].$$

Next we discuss the application of a sharp version of the non-diagonalizable Bauer–Fike theorem. Dai [30] reveals the fact that a stable homogeneous singular system

$$B\dot{x} = Ax, \tag{5.3}$$

where $x \in \mathbb{R}^n$ and $\mathrm{rank}(B) < n$ may not be structurally stable. The following proposition gives the condition of structurally stability of a stable homogeneous singular system.

**Proposition 5.4** *([30, Proposition 4.1]). Let the system (5.3) be stable, i.e.,* $\sigma(\{A, B\}) \subset \mathbb{C}^-$ *(eigenvalues in the open left complex plane). Then (5.3) is structurally stable if and only if*

$$\deg(\det(\lambda B - A)) = \mathrm{rank}(B), \tag{5.4}$$

*where* $\deg(\cdot)$ *means the degree of a polynomial.*

It follows from (5.4) that the matrix pair $\{A, B\}$ must be regular, if the matrix pair $\{A, B\}$ is singular [6,11], i.e., $\det(\lambda B - A) \equiv 0$, then the matrix $B$ is degenerative. We can use the above result to give a simple perturbation bound of structurally stable homogeneous singular system.

The problem is given as follows. Consider the homogeneous singular system (5.3). Assume that $\sigma(\{A, B\}) \subset \mathbb{C}^-$ and (5.4) holds. Next we consider which condition of the perturbation matrix $\delta A$ should be satisfied to preserve the stability, i.e., $\sigma(\{A + \delta A, B\}) \subset \mathbb{C}^-$. Dai [30, Theorem 4.1] gives a complicated result, we manage to give another version.

**Theorem 5.5.** *Consider the structurally homogeneous singular system, if*

$$\|P^{-1}\delta AQ^{-1}\|_p \le \frac{\Delta^m}{1 + \Delta + \Delta^2 + \cdots + \Delta^{m-1}}, \quad (p = 1, \infty) \tag{5.5}$$

*then the stability is preserved, where* $\Delta = \min_{i\in\langle n\rangle}\{-\mathrm{Re}(\lambda_i)|\lambda_i \in \sigma(\{A, B\})\}$, $\mathrm{Re}(\lambda_i)$ *is the real part of* $\lambda_i$, $m$, $P$, *and* $Q$ *are the same as Eq. (4.1).*

**Proof.** Applying Corollaries 4.4 and 4.5, when $\delta B = 0$, we can get

$$\min_{i\in\langle n\rangle} \frac{|\lambda_i - \mu|^m}{1 + |\lambda_i - \mu| + |\lambda_i - \mu|^2 + \cdots + |\lambda_i - \mu|^{m-1}} \le \|P^{-1}\delta AQ\|_p.$$

Because of the function $f(x) = \frac{x^m}{1+x+x^2+\cdots+x^{m-1}}$ is increasing when $x > 0$, if $\|P^{-1}\delta AQ\|_p \le \frac{\Delta^m}{1+\Delta+\Delta^2+\cdots+\Delta^{m-1}}$, $|\lambda_i - \mu| \le \Delta$, $\mathrm{Re}(\mu) < 0$, then the stability is preserved. $\square$

## 6. Concluding remarks

We present sharp versions of Bauer–Fike's theorem for the diagonalizable and non-diagonalizable matrices, which we replace the matrix norm with its spectral radius or sign-complex spectral radius, 1-norm and $\infty$-norm, respectively. If the matrix $A$ is invertible, then we can provide the relative error bound. It is natural to ask if we can extend our results to the singular case [31–33] for the sharp relative error bound, which will be our future research topic.

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